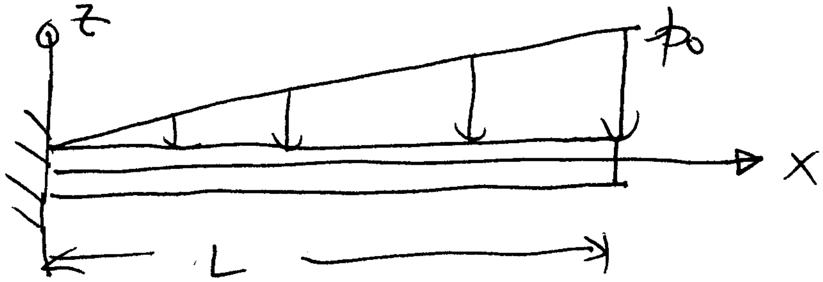
PAL
2/28/08

Unified Engineering Problem Set
Week #5 Spring 2008

SOLUTIONS

M 5.1

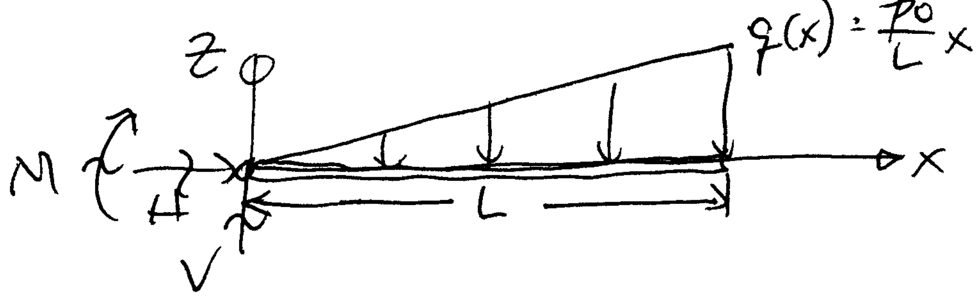
(a) First work the original configuration:



Loading: $q(x) = \frac{p_0}{L} x$

To determine the deflection, $w(x)$, need the moment, $M(x)$.

Start with the free body diagram:



Apply equilibrium:

$$\sum F_x = 0 \ \overset{+}{\rightarrow} \Rightarrow H = 0$$

$$\sum F_z = 0 \ \uparrow + \Rightarrow V - \int_0^L \frac{p_0}{L} x \, dx = 0$$

gives: $V = \frac{p_0}{2L} x^2 \Big]_0^L$

$$V = \frac{p_0 L}{2}$$

$$\sum M_0 = 0 \ \overset{+}{\circlearrowleft} \Rightarrow -M - \int_0^L \frac{p_0}{L} x^2 dx = 0$$

gives:

$$M = -\frac{p_0 x^3}{3L} \Big]_0^L$$

$$M = -\frac{p_0 L^2}{3}$$

Now proceed via:

$$S(x) = \int q(x) \, dx$$

$$\Rightarrow S(x) = -\int \frac{p_0}{L} x \, dx$$

$$S(x) = -\frac{p_0 x^2}{2L} + C_1$$

$S(x) = V$ @ $x = 0$

$$\Rightarrow \frac{p_0 L}{2} = C_1$$

gives: $S(x) = \frac{p_0}{2} \left( L - \frac{x^2}{L} \right)$

Check: at the tip $(x=L)$, $\mathcal{S}=0$

$$\mathcal{S}(L)=\frac{p_0}{2}\left(L-\frac{L^2}{L}\right)=0 \quad \checkmark \ \underline{\text{ok}}$$

Proceed to:

$$M(x)=\int \mathcal{S}(x)\,dx$$

$$=\frac{p_0}{2}\int\left(L-\frac{x^2}{L}\right)dx$$

$$\Rightarrow M(x)=\frac{p_0}{2}\left(Lx-\frac{x^3}{3L}\right)+C_2$$

$M(x)=M$ @ $x=0$

$$\Rightarrow -\frac{p_0 L^2}{3}=C_2$$

gives:

$$M(x)=\frac{p_0}{2}\left(Lx-\frac{x^3}{3L}-\frac{2L^2}{3}\right)$$

Check: again at the tip $(x=L)$, $M=0$

$$M(L)=\frac{p_0}{2}\left(L^2-\frac{L^2}{3}-\frac{2L^2}{3}\right)=0 \quad \checkmark \ \underline{\text{ok}}$$

Proceed to:

$$M=EI\frac{d^2w}{dx^2}$$

$$\Rightarrow \frac{d^2w}{dx^2}=\frac{p_0}{2EI}\left(Lx-\frac{x^3}{3L}-\frac{2L^2}{3}\right)$$

First integral gives:

$$\frac{dw}{dx} = \frac{p_0}{2EI}\left(\frac{Lx^2}{2} - \frac{x^4}{12L} - \frac{2L^2x}{3}\right) + C_3$$

Second integral gives:

$$w = \frac{p_0}{2EI}\left(\frac{Lx^3}{6} - \frac{x^5}{60L} - \frac{L^2x^2}{3}\right) + C_3x + C_4$$

We need two boundary conditions on displacement. For a clamped boundary, both the slope and displacement are zero:

$$@\ x = 0, \quad w = 0 \text{ and } \frac{dw}{dx} = 0$$

$$\Rightarrow \quad C_3 = 0, \quad C_4 = 0$$

Finally:
$$w(x) = \frac{p_0}{2EI}\left(\frac{Lx^3}{6} - \frac{x^5}{60L} - \frac{L^2x^2}{3}\right)$$

can simplify a bit:

$$\boxed{w(x) = \frac{p_0 x^2}{6EI}\left(\frac{Lx}{2} - \frac{x^3}{20L} - L^2\right)}$$

For a cantilevered configuration the maximum will be at the tip: $x = L$

so:

$$w_{max} = \frac{p_0 L^2}{6EI}\left(\frac{L^2}{2} - \frac{L^2}{20} - L^2\right)$$

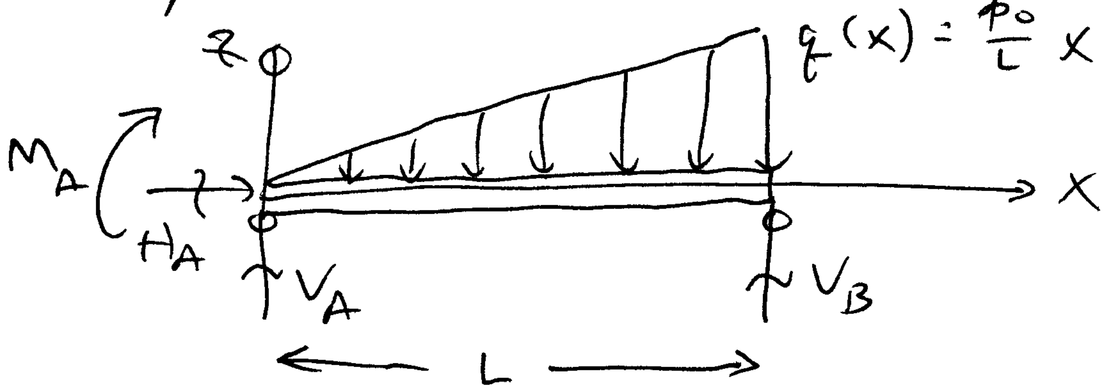
gives:
$$w_{max} = -\frac{p_0 L^4}{6EI}\left(\frac{10}{20} - \frac{1}{20} - \frac{20}{20}\right)$$

$$\Rightarrow \boxed{w_{max} = \frac{-11 p_0 L^4}{120 EI}}$$

Checks:

- negative ... downward ... consistent with loading
- units: $\dfrac{\left[\frac{F}{L}\right]\left[L^4\right]}{\left[\frac{F}{L^2}\right]\left[L^4\right]} = [L]$ ✓ ok

→ Now add the roller support at the tip and redraw the free body diagram:



The difference is that this configuration is Statically Indeterminate.

The basic procedure is the same, but the equations need to be solved simultaneously rather than in sequence.

Proceeding:

$$\sum F_x = 0 \quad \xrightarrow{+} \quad \Rightarrow H_A = 0 \quad \text{again} \qquad (1)$$

$$\sum F_z = 0 \quad \uparrow + \quad \Rightarrow V_A + V_B - \int_0^L \frac{p_0}{L} x \, dx = 0$$

gives: $$V_A + V_B = \frac{p_0 L}{2} \qquad (2)$$

$$\sum M_A = 0 \quad \circlearrowleft + \quad \Rightarrow -M_A + V_B L - \int_0^L \frac{p_0}{L} x^2 dx = 0$$

gives:
$$-M_A + V_B L = \frac{p_0 L^2}{3} \qquad (3)$$

We move on to getting the distribution of the Shear and Moment Resultants

use: $$S(x) = \int q(x) \, dx$$

and again get:
$$S(x) = -\frac{p_0 x^2}{2L} + C_1$$

However, in this case the boundary condition is not yet determined. We know @ $x = 0$, $S = V_A$, so express this in terms of the yet-determined resultant:

$$C_1 = V_A$$

So:

$$S(x) = -\frac{p_0 x^2}{2L} + V_A \qquad (4)$$

Proceed to:

$$M(x) = \int S(x)\,dx$$

$$\Rightarrow M(x) = -\frac{p_0 x^3}{6L} + V_A x + C_2$$

Again, the boundary condition is that:

$$M(x) = M_A \quad @ \; x = 0$$

$$\Rightarrow C_2 = M_A$$

giving:

$$M(x) = -\frac{p_0 x^3}{6L} + V_A x + M_A \qquad (5)$$

Proceed to the Moment-Curvature Relation of:

$$M = EI \frac{d^2 w}{dx^2}$$

$$\Rightarrow \frac{d^2 w}{dx^2} = \frac{1}{EI}\left\{-\frac{p_0 x^3}{6L} + V_A x + M_A\right\}$$

The first derivative gives:

$$\frac{dw}{dx} = \frac{1}{EI}\left\{ -\frac{p_o x^4}{24L} + V_A \frac{x^2}{2} + M_A x \right\} + C_3$$

again we have $\frac{dw}{dx} = 0$ @ $x = 0$

$$\Rightarrow C_3 = 0$$

The next derivative gives:

$$w = \frac{1}{EI}\left\{ \frac{-p_o x^5}{120L} + V_A \frac{x^3}{6} + M_A \frac{x^2}{2} \right\} + C_4$$

and again we have $w = 0$ @ $x = 0$

$$\Rightarrow C_4 = 0$$

We now have one other boundary condition on displacement to apply and this will allow us to begin to work the developed equations.
At the roller support ($x = L$), the displacement also is 0. Using this:

$$w(L) = 0 = -\frac{p_o L^4}{120} + V_A \frac{L^3}{6} + M_A \frac{L^2}{2}$$

Work this to:

$$-\frac{p_o L^2}{60} + \frac{V_A L}{3} + M_A = 0 \qquad (6)$$

We now have equations (2), (3), and (6) to determine $V_A$, $V_B$, and $M_B$.

from (2): $$V_B = \frac{p_0 L}{2} - V_A \qquad (2')$$

and using in (3):

$$-M_A + \frac{p_0 L^2}{2} - V_A L = \frac{p_0 L^2}{3}$$

$$\Rightarrow M_A = \frac{p_0 L^2}{6} - V_A L \qquad (3')$$

Finally using this in (6):

$$-\frac{p_0 L^2}{60} + \frac{V_A L}{3} + \frac{p_0 L^2}{6} - V_A L = 0$$

working:

$$\frac{2}{3} V_A = p_0 L \left( \frac{10}{60} - \frac{1}{60} \right)$$

so: $$\boxed{V_A = \frac{9}{40} p_0 L}$$

using in (2'): $$\boxed{V_B = \frac{11}{40} p_0 L}$$

and in (3'): $$\boxed{M_A = -\frac{7}{120} p_0 L^2}$$

We return to the equation developed for w(x):

$$w(x) = \frac{1}{EI}\left\{ -\frac{p_0 x^5}{120L} + V_A \frac{x^3}{6} + M_A \frac{x^2}{2} \right\}$$

and use the determined values:

$$w(x) = \frac{1}{EI}\left\{ -\frac{p_0 x^5}{120L} + \frac{9 p_0 L x^3}{240} - \frac{7 p_0 L^2 x^2}{240} \right\}.$$

Pull out a common factor and also express this in the parameter $x/L$ so this varies from 0 to 1 along the beam:

$$w(x) = \frac{p_0 L^4}{240 EI}\left\{ -2\left(\frac{x}{L}\right)^5 + 9\left(\frac{x}{L}\right)^3 - 7\left(\frac{x}{L}\right)^2 \right\}.$$

Let's check that we have the right units. Everything in parentheses is dimensionless, so as in the previous case:

$$\frac{[F/L][L^4]}{[F/L^2][L^4]} = [L] \quad \checkmark \quad \underline{\underline{Yes}}$$

To find the point of maximum deflection (magnitude thereof) and its location, take the derivative and set it to zero. In this case we know that the maximum <u>will not</u> occur at the boundaries since the deflection is zero at both locations. Thus:

$$\frac{dw}{dx} = 0 = -10\left(\frac{x}{L}\right)^4 + 27\left(\frac{x}{L}\right)^2 - 14\left(\frac{x}{L}\right)$$

This does occur @ $x = 0$, but we know it must and that is where $w = 0$. Look for this being satisfied between $\frac{x}{L} = 0$ to 1. One can explicitly solve the quartic or try values to work to the answer

→ I choose the latter. I know the maximum deflection will occur somewhere near the middle and probably closer to the roller since this is the less stiff and more highly loaded end. I start with $\frac{x}{L} = 0.6$

$$\frac{x}{L} = 0.6 \text{ gives } 0.024$$

<u>not a bad guess!</u>

This is a positive slope and from the problem I see that the deflection will look like:

So I need to go back from this point.

Try $\frac{x}{L} = 0.55 \Rightarrow -0.448$

So I overshot.... try $\frac{x}{L} = 0.59 \Rightarrow -0.073$

So my first choice was very good (or I was just lucky....). The two decimal points are within problem definition so use:

$$\frac{x}{L} = 0.40$$

Plug this into the equation for $w(x)$:

$$\boxed{w_{max} = -0.00305 \frac{p_0 L^4}{EI} \quad \text{at } \frac{x}{L} = 0.40}$$

Compare the two cases.....

Without the roller support, the beam has a maximum deflection of $-0.092 \frac{p_0 L^4}{EI}$ which is reduced to $-0.00305 \frac{p_0 L^4}{EI}$ with the roller -- on the order of a factor of 30. The support at the tip is especially important since the load is highest there.

Furthermore, the moment carried at the clamped end is substantially reduced from $-\frac{1}{3} p_0 L^2$ to $-0.058\, p_0 L^2$. The vertical resultants are distributed between the two supports in the second case and the roller provides a moment balance through the moment arm over which it acts.